Reversible and irreversible sources of radiation entropy

RICHARD GOODY and WEDAD ABDOU
Jet Propulsion Laboratory, USA

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SUMMARY

We discuss two practical issues concerning the relationship between the entropies of the atmosphere and of the radiation field. The first issue is whether the radiative flux of entropy can be determined from a satellite with sufficient accuracy for climate studies. We conclude that an accuracy much better than 1 % is required, but that this can be achieved, in the thermal spectrum, with spectrometers having spectral resolutions $\sim 1\ \mathrm{cm}^{-1}$ It is also possible that the required accuracy can be achieved with a non-spectral approach. The second issue is whether the rate of change of the atmospheric entropy inventory can be inferred from the outgoing radiative entropy flux. We conclude that, for clear skies, and in the thermal region of the spectrum, the two are directly proportional, and that the atmospheric term can be inferred with useful accuracy from an empirical relation.

1 Introduction

The two decades that have elapsed since the seminal paper by Paltridge (1975) have shown a steady rise in interest in the entropy of the atmosphere and the entropy carried on the radiation field. There is, however, no direct relationship between these two quantities. The interesting quantity is the entropy of the atmosphere, but it is difficult to measure, while the radiation flux to space is simple to measure, but more difficult to interpret.

We may treat the atmosphere and the radiation field as two separate, interacting systems, with entropy and internal energy per unit volume equal to s^a and e^a for the atmosphere and to s^r and e^r for the radiation field. For the combined system,

$$s = s^a + s^r, \tag{1}$$

$$e = e^a + e^r. \tag{2}$$

If matter and radiation interact in a closed system, then, according to the first law of thermodynamics,

$$\dot{e} = \dot{e}^a + \dot{e}^r = 0, \tag{3}$$

while, according to the second law of thermodynamics,

$$\dot{s} = \dot{s}^a + \dot{s}^r \geq 0. \tag{4}$$

The dots indicate rates of change. The inequality refers to irreversible changes.

Dynamical meteorology treats the atmosphere as a fluid that is in a state of local thermodynamic equilibrium, with a well-defined value of the local temperature, T. Such is consistent only with reversible heat interactions with the atmosphere. Hence, neglecting radiative work interactions on the atmosphere[1],

$$\dot{s}^a = \frac{\dot{e}^a}{T}. \tag{5}$$

It follows from (3), (4) and (5) that,

$$\dot{s}^r \geq \frac{\dot{e}^r}{T}. \tag{6}$$

Irreversible changes are involved in the process of radiative transfer[2], as we may see from the following argument. The equation of radiative transfer allows us to separate $\dot{e}^r$ into two terms, the energy change by absorption, $\dot{e}^r_{\text{abs}} \leq 0$ and the energy change by emission, $\dot{e}^r_{\text{emit}} \geq 0$. For the sake of this example, assume that there is no net change in the energy of the radiation field, or, from (3), in the energy of the atmosphere,

$$\dot{e}^r = \dot{e}^r_{\text{emit}} + \dot{e}^r_{\text{abs}} = 0. \tag{7}$$

From (3) and (5) it also follows that the atmospheric energy and entropy do not change.

Let the matter and radiation have temperatures T and T_{rad}, respectively. These temperatures will differ except inside a constant-temperature enclosure. The emitted component originates in the matter and will create entropy at a rate $\dot{e}^r_{\text{emit}}/T$, while the absorbed component does so at a rate $\dot{e}^r_{\text{abs}}/T_{\text{rad}}$. The net interchange of energy is zero, and reversible entropy changes are also zero. Consequently the total entropy source (for this simple example) consists of these two terms only, and is irreversible.

$$\begin{aligned} \dot{s}^r &= \frac{\dot{e}^r_{\text{abs}}}{T_{\text{rad}}} + \frac{\dot{e}^r_{\text{emit}}}{T}, \\ &= \dot{e}^r_{\text{emit}} \left(\frac{1}{T} - \frac{1}{T_{\text{rad}}} \right). \end{aligned} \tag{8}$$

[1] Mechanical work is of the order of v/c times the change of internal energy, where v is the velocity of the atmosphere, and c is the velocity of light

[2] "An irreversible element is introduced by the addition of emitting and absorbing substance", Max Planck (1959), p.190.

We shall refer to (8) as the *irreversible source of radiation entropy.* In the next section we shall show that (8) is positive definite. The irreversible process involved is the thermalisation of radiative energy. It affects the radiation entropy, but not the atmospheric entropy, and is not relevant to the behavior of the atmosphere itself. Measurements of the entropy of the radiation field will be affected by this source, and it must be accounted for it we are to separate out the reversible sources.

The irreversible source is particularly important for solar radiation for which $T_{\rm rad} \approx 6000$ K. For solar absorption followed by thermal emission the rate of increase of irreversible entropy is large and can greatly exceed any increase of atmospheric entropy. To seek a direct relationship between the entropy of solar radiation and atmospheric entropy is not likely to be rewarding.

Useful relationships may be shown to exist for thermal radiation, however, and we shall restrict the discussion to this topic. For these relationships to be of practical value, we need to know whether entropy measurements of sufficient accuracy can be made from a satellite. The principal source of error is the finite spectral resolution of the observing instrument, The first objective of this paper is to understand this limitation,

From a satellite we can measure the fluxes of radiant energy and radiant entropy leaving the atmosphere. Fluxes are related to source terms try (see Goody and Yung 1989, Eq.2.11),

$$\frac{\partial F(s)}{\partial z} = \dot{s}^r, \tag{9}$$

$$\frac{\partial F(e)}{\partial z} = \dot{e}^r \tag{10}$$

and if the lower limits to the integrals are inside the lower surface, the fluxes to space are,

$$F_\infty(s) = \int_0^\infty \dot{s}^r \, dz, \tag{11}$$

$$F_\infty(e) = \int_0^\infty \dot{e}^r \, dz \tag{12}$$

The right sides of Eqs.(11) and (12) are the column inventories of $\dot{s}^r$ and $\dot{e}^r$, respectively. Before concluding this section we may ask why such inventories are relevant to current research. There are three reasons. First, both energy and entropy are state functions and their inventories help to define the state of the climate system, see the work of Peixoto *et al.* (1991). Second, according to a well-known theorem,

$$\int_V \dot{s}^a \, dV = -\int_V \frac{\dot{\phi}_{\rm irr}}{T} \, dV, \tag{13}$$

where ϕ_{irr} is the irreversible fluid dissipation and V represents the climate system. If the left side of (13) can be derived from the radiative entropy flux, we have a measure of an important constraint on climate models. '1'bird, according to Paltridge (1975), the state of the atmosphere corresponds to a minimum in the left side of (13). While Paltridge's conjecture has not been proved, it has stimulated interesting discussions.

2 Sources of radiation entropy

'he relationship between energy radiance, I_ν, and entropy radiance, L_ν, was first established by Planck but was derived in more modern terms by Rosen (1954),

$$\frac{c^2 L_\nu}{2k\nu^2} = \left(\frac{c^2 I_\nu}{2h\nu^3}+1\right)\ln\left(\frac{c^2 I_\nu}{2h\nu^2}+1\right) - \left(\frac{c^2 I_\nu}{2h\nu^3}\right)\ln\left(\frac{c^2 I_\nu}{2h\nu^3}\right). \tag{14}$$

In Eq.(14), c, h, ν, k all have their usual meanings.

Equation (14) expresses a one-to-one relationship between energy and entropy radiances. A stream of monochromatic radiation is a onc-parameter system and, according to a fundamental theorem in thermodynamics, there is no distinction between the first and second laws for one- and two-parameter systems. This draws attention to the fact that thei c is no mere information in the entropy radiances than in the energy radiances. Consequently, if we work with observed entropy radiances, we can learn nothing more in principle than from energy radiances. However, the information is organized in a different way, and that can be important, in practice.

Radiative source terms, per unit volume, are (see Goody ancl Yung 1989, Eq.2.11),

$$\dot{s}^r = \int_{4\pi} d\omega_l \int_0^\infty d\nu \frac{dL_\nu}{dl}, \tag{15}$$

$$\dot{e}^r = \int_{4\pi} d\omega_l \int_0^\infty d\nu \frac{dI_\nu}{dl}. \tag{16}$$

Differentiating (14) with respect to displacement in the l-direction gives,

$$\begin{aligned}\frac{dL_\nu}{dl} &= \frac{k}{h\nu}\ln\left(\frac{1+\frac{c^2 I_\nu}{2h\nu^3}}{\frac{c^2 I_\nu}{2h\nu^3}}\right)\frac{dI_\nu}{dl}, \\ &= \frac{1}{T_\nu}\frac{dI_\nu}{dl}.\end{aligned} \tag{17}$$

T_ν is the emission temperature, obtained by replacing B_ν (the Planck function) by I_ν in the expression for the radiance for complete thermodynamic equilibrium,

$$I_\nu = \frac{2h\nu^3/c^2}{\exp(h\nu/kT_\nu) - 1} \tag{18}$$

Complete thermodynamic equilibrium (as contrasted with local thermodynamic equilibrium) exists inside a constant-temperature enclosure. Then, from Kirchoff's laws,

$$I_\nu = B_\nu, \tag{19}$$

and T_ν becomes equal to the enclosure temperature T. For complete thermodynamic equilibrium, according to (4),

$$\dot{s}^r + \dot{s}^a = 0. \tag{20}$$

In general, $T \neq T_\nu$, and $\dot{s}^r + \dot{s}^r \neq 0$. We may write, formally,

$$\dot{s}^r + \dot{s}^a = \dot{s}^r_{\mathrm{irr}}. \tag{21}$$

$\dot{s}^r_{\mathrm{irr}}$ is the irreversible contribution to the change in the entropy of the radiation field. According to the second law of thermodynamics it is positive.

This result may be confirmed follows. From Eqs.(5), (15), (16), and (21) we may write,

$$\dot{s}^r_{\mathrm{irr}} = \int_{4\pi} d\omega_l \int_0^\infty \left(\frac{1}{T_\nu} - \frac{1}{T}\right) \frac{dI_\nu}{dl}\, d\nu. \tag{22}$$

For local thermodynamic equilibrium, Schwarzschild's equation of transfer for thermal radiation is,

$$\frac{dI_\nu}{dl} = e_\nu(B_\nu - I_\nu), \tag{23}$$

where e_ν is the extinction coefficient. Hence,

$$\dot{s}^r_{\mathrm{irr}} = \int_{4\pi} d\omega_l \int_0^\infty e_\nu(B_\nu - I_\nu)\left(\frac{1}{T_\nu} - \frac{1}{T}\right) d\nu. \tag{24}$$

From the definition of T_ν by Eq. (18), we may show that,

$$\frac{1}{T_\nu} - \frac{1}{T} = \frac{k}{h\nu} \ln\left\{1 + \frac{B_\nu - I_\nu}{I_\nu(c^2 B_\nu/2h\nu^3 + 1)}\right\} \tag{25}$$

The sign of (25) is the same as for $(B_\nu - I_\nu)$, and (24) is, therefore, positive definite.

3 Calculation of sources

We obtained profiles of temperature from a one- dimensional radiative-tropical-convection model, kindly loaned to us by Dr. Arthur Hou. The model is specified by the solar input and a relationship between temperature and humidity. Details of the model are unimportant for our purposes; it simply provides atmospheric profiles that resemble climate states.

Energy radiances are calculated with the AFGL MODTRAN program. Throughout the thermal spectrum, MODTRAN has a spectral resolution of about 1 cm^{-1}. Entropy radiances are calculated from energy radiances using Eq.(14) (for the importance of the finite spectral resolution of MOTRAN, see the following section). Calculations are made for 34 discrete levels. The calculation stops 34 km above the surface. MODTRAN includes an elementary cloud model, and gaseous densities are variables.

At each level we calculate fluxes of energy and entropy,

$$F(e) = \int_0^\infty d\nu \int_{-1}^{+1} 2\pi I_\nu(\xi)\xi\, d\xi, \tag{26}$$

$$F(s) = \int_0^\infty d\nu \int_{-1}^{+1} 2\pi L_\nu(\xi)\xi\, d\xi, \tag{27}$$

where ξ is the cosine of the zenith angle.

The source terms are calculated from finite differences,

$$\dot{s}_n^r = \frac{F_{n+1}(s) - F_n(s)}{z_{n+1} - z_n}, \tag{28}$$

$$\dot{e}_n^r = \frac{F_{n+1}(e) - F_n(e)}{z_{n+1} - z_n}, \tag{29}$$

$$\dot{s}_n^a = -\frac{\dot{e}_n^r}{(T_{n+1} + T_n)/2}. \tag{30}$$

In the subsequent discussion we use the numerical data for comparative purposes only, and systematic errors, such as from the limited spectral resolution, tend to cancel. We do not believe that any of our conclusions are affected by the accuracy of the numerical algorithms.

In a recent paper Li, Chýlek and Lesins (1994) have calculated entropy source terms for a radiative-convective model. They use two different procedures. The method they used for solar radiation corresponds to the straightforward use of Eq.(14), in the manner of this paper. For thermal radiation they develop an equation of transfer for entropy radiance, and approximate the source function by assuming complete thermodynamic equilibrium. We discuss this approach in the appendix. It is in error for optical depths that are not large.

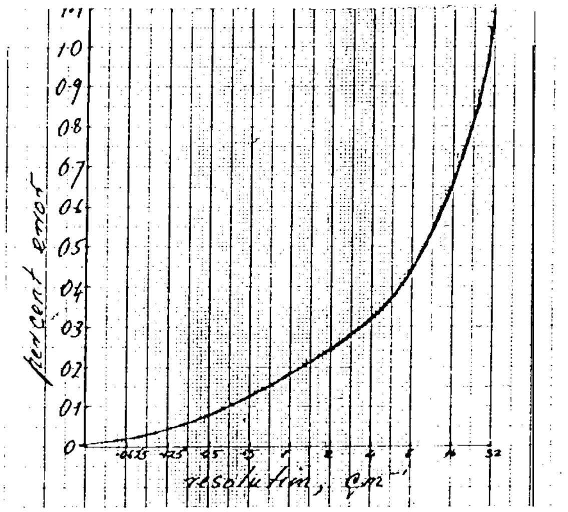

Figure 1: Errors in entropy calculations caused by spectral averaging.

4 Effect of finite spectral resolution

Equation (14) is non-linear, and the calculated entropy radiance will depend upon the degree of averaging (the spectral resolution) of the energy radiances. Outgoing radiances from the earth's atmosphere show a finest ructure of rotation lines that cannot be completely resolved by misting satellite spectrometers. Some information is lost in the process of averaging, and this reflects itself in a positive error in the entropy calculation.

How precisely do we need to calculate entropy fluxes? Lesins (1990) shows that important climate issues involve global average entropy changes of about 1%. There are questions involving absolute *versus* relative precision, and there is the possibility that averaging data from different locations may lead to global data of higher precision. Nevertheless, Lesins' figure offers one yardstick for judging the possible impact of computational errors. Ideally we would like to see entropy errors not greater than 0.1%.

The AIRS spectrometer that is to be flown on the second EOS satellite, can record from 564 to 2964 cm^{-1} with a spectral resolution of approximately 1 cm^{-1}. This resolution can also be achieved by other satellite observing systems. An outgoing spectrum of the energy radiance for this spectral region was available to us at resolutions between 0.002 and 0.004 cm-1, infinite resolution for all practical purposes. Entropy radiances were calculated from Eq.(14) and integrated over the entire spectral range. This was done first at full resolution, and then repeated after smoothing the data over intervals from 0.0625 to 32 cm-1. The nonlinear character of (14) is such that averaging increases the entropy.

The percentage error as a function of spectral resolution is shown in Fig.1. The error is 0.18% for 1 cm^{-1} resolution and 0.1% for 0.35 cm^{-1} resolution. The latter resolution is technically achievable; errors from limited spectral resolution can probably be held within the desired limits. The slow increase of the error with resolution below 4 cm^{-1} is attributable to gradual smoothing out of the rotational fine structure. When the smoothing is 8 cm^{-1} or greater serious errors appear. These errors arc associated with smoothing of the gross structure of the bands.

Wc made similar calculations of the effect of spectral resolution on the entropy calculation for four sensitive regions near to band centers. Fractional errors in the center of the 9.5μm O_3 band were several times larger than those shown in Fig.1 but, averaged over the entire spectrum, they do not appear to matter.

An interesting feature of this discussion is that both Lesins (1990) and Stephens and O'Brien (1993) calculate entropy fluxes at the top of the atmosphere using energy radiances integrated over the entire spectrum, derived from low-resolution ERBE data. The basis for their calculation is an assumption that the outgoing radiance can be approximated by the radiance from a black body at the emission temperature, T_e. If the outgoing radiance is the Planck function, it may be shown that,

$$F^{+,-}(e) = \sigma T_e^4, \tag{31}$$

$$F^{+,-}(s) = \frac{4}{3}\sigma T_e^3. \tag{32}$$

(31) is Stefan's law. Since T_e is calculated from the total energy flux, this relation ensures that the energy flux used in the approximation is identical to the actual energy flux. (32) is the result of integrating (14) over frequency, using the Planck function, see Essex (1 984). The superscripts (+, -) indicate that the integrals in (26) and (27) are taken over the two hemispheres independently. T_e will differ for the two hemispheres.

If wc eliminate T_e between (31) and (32), we find,

$$F^{+,-}(s) = 2.0575 \times 10^{-2}\{F^{+,-}(e)\}^{\frac{3}{4}}, \text{W 111-21\{-” ‘}. \tag{33}$$

Since the emission temperature dots not appear in Eq.(33) it is conjectured that it may be approximately correct for non-equilibrium radiation fields.

To investigate the errors involved in Eq. (33) wc have calculated fluxes directly using MODTRAN together with Eq.(14), for a variety of atmospheres with different solar fluxes and cloud amounts. As we have Shown) entropy fluxes calculated with 1 cm^{-1} resolution may be too large by about 0.2%, but to calculate at maximum spectral resolution would have involved unreasonable amounts of computing. The relationship between $F(s)$ and $F(e)^{\frac{3}{4}}$ at all levels,

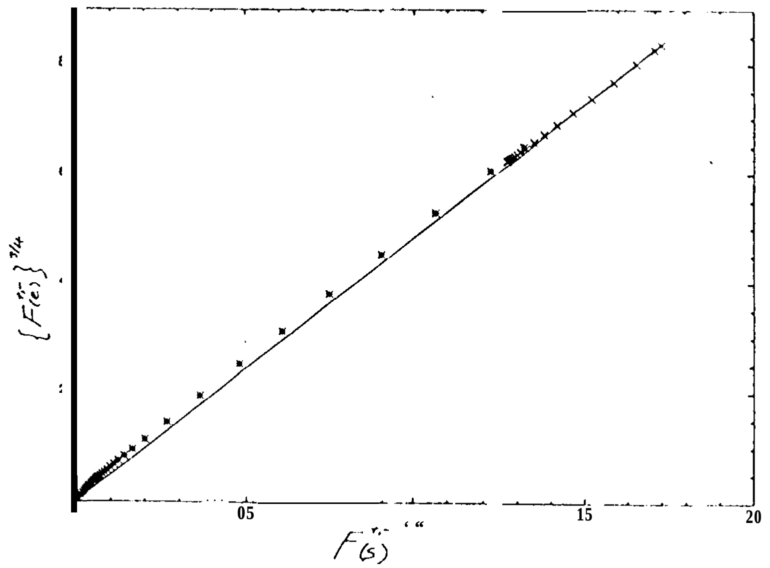

Figure 2: Test of Eq. (33) for one atmospheric structure. Crosses (×) are upwelling fluxes; asterisks (∗) are downwelling fluxes. The full line follows Eq.(33). Both axes are in MKS units. For this ascent the upwelling entropy flux at the top of the atmosphere was 1.271 W m^{-} K^{-1}.

for one of these atmospheres, is shown in Fig. 2. Agreement with Eq.(33) is remarkably good for upward flux components; less so for downward components.

For the purposes of this paper, the important quantity is tile upwelling flux at the top of the atmosphere. On the basis of 20 different ascents, with solar fluxes between 200 and 400 W m^{-2}, some with clouds and some without clouds, wc found a least-squares best fit,

$$F_\infty(s) = 2.043 \times 10^{-2} F_\infty(e)^{\frac{3}{4}} + 2.41 \times 10^{-3}, \mathrm{W\,m^{-2}K^{-1}}. \quad (34)$$

The zero offset in Eq.(34) is not significant. The root- mean-square deviation from Eq.(34), as a percentage of the mean value of $F_\infty(s)$ (1. 193 W m^{-2} K- ^{1}), is 0.3570. This error is surprisingly small in view of the fact that no information associated with the line and band structure is used. The numerical factor 2.043 in Eq. (34) should be modified to 2.039 to allow for errors associated with 1 cm^{-1} averaging.

5 Entropy sources for clear skies

Equations (28), (29), and (30) were evaluated using MODTRAN. The differencing involved in Eqs.(28) and (29) means that resolution errors should be unimportant for the source terms. The irreversible term in the radiation entropy was calculated from Eq.(21). Wc first discuss 7 cases of cloud-free conditions.

Figure 3 shows the results for onc clear-sky ascent. 'he irreversible entropy sour-cc is always positive definite, and smaller in magnitude than the atmospheric and radiation source terms. We arc interested in inventories of these quantities, scc Eqs. (1 1), (1 2), and (13). These involve integrals through the atmosphere, including surface terms. These surface terms are shown in the insert to Fig.3. The question at issue is whether the inventory of $\dot{s}^a$ can be inferred from the inventory of .4", which, from (1 1), is equal to the entropy flux to space. A least-squares best fit to the data gave tile following relation between these two quantities,

$$\int_0^{cm} s" \, dz = 0.7376 \ \mathrm{Fro(s)} + 2.04 \ \mathrm{X} \ 10^{-4}, \mathrm{W\,m^{-2}K^{-1}}. \quad (35)$$

The zero offset is not statistically significant. The root.-lnean-squarc deviation of data from Eq.(35), expressed as a percentage of the me-au value of $\int_0^\infty \dot{s}^a dz$ (0.9358 W $m^{-2}K^{-1}$), is 0.17%.

Wc had anticipated a relation of the nature of (35). It is known (see, for example, Goody and Yung 1989, §6.4.1) that the radiation-to-space approximation works remarkably well at all levels in a cloud-free atmosphere. This means that most cooling is by photons that escape to space. For this process the irreversible term is known precisely. A photon escaping to space can potentially do work at a rate that combines a very small radiation pressure with the very

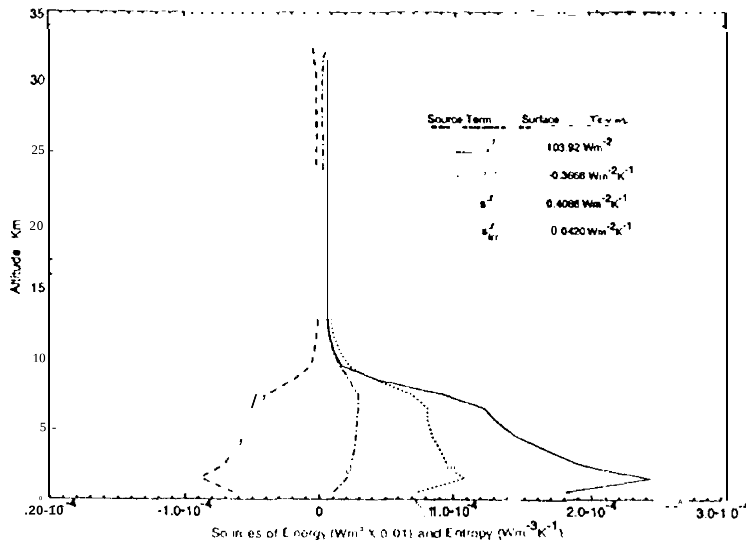

Figure 3: Source terms. 'his calculation is for a clear sky, for a solar flux of 300 W m^{-2}, and the Manabe-Wetherald relative humidity climatology. For $\dot{s}^r$ and $\dot{e}^r$ the surface term is a radiation flux.

large velocity of light, and is one third of the rate of escape of internal energy. Since this work is not used by the atmosphere[3], there is an increase of entropy. This accounts for the factor $\frac{4}{3}$ in Eq.(32). Thus, the anticipated value of the slope in Eq.(35) is 0.75. Some irreversible thermalization is also involved, but it is small, and is satisfactorily included empirically.

This combination of a theoretical basis, together with statistical errors less than 0.270, shows that changes in the inventory of atmospheric entropy can be inferred from outgoing radiation entropy fluxes to a degree of accuracy that is useful for climate calculations.

6 Entropy sources for cloudy skies

For satellite fields of view that include both cloudy and clear areas, algorithms have been developed that yield the clear-sky radiances (Chahine 1976). nom such radiances the entropy inventory can be obtained by usc of Eq.(35).

What can be said about the cloud-covered regions? Remote sensing at optical wavelengths is not appropriate for below-cloud data in cloud- covered regions, whether they be entropies or any other state variables. This is not such a crucial

[3] Radiation from the sun can perform work on the solar sail of a fast, interstellar spacecraft.

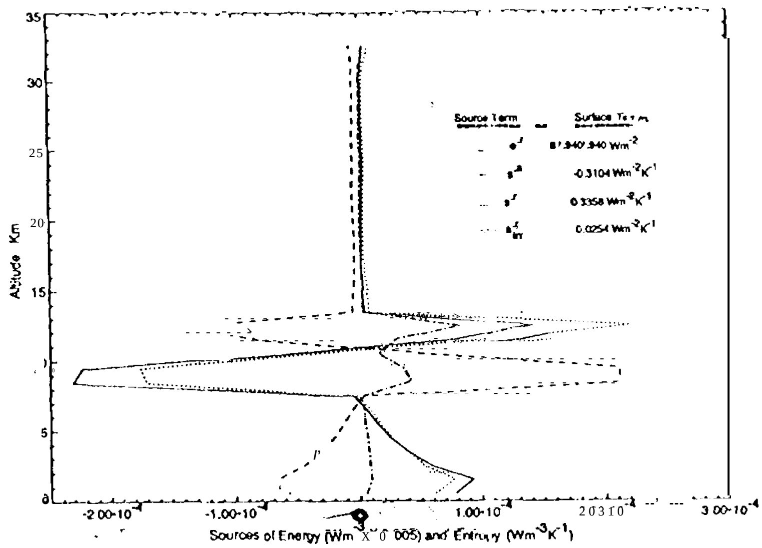

Figure 4: Source terms for a cloudy atmosphere. The cloud lies between 8 and 12 km.

limitation as might appear at first sight. A comparison between observation and prediction in cloud-free regions is a good test of a numerical model, even though incomplete.

13 cases of cloudy atmospheres were calculated, and for these the slope in Eq.(35) varied from 0.4311 to 0.7353. For some cases of low clouds, (35) may be of value, but it is difficult to select the appropriate cases *a priori.*

Figure 4 shows entropy and energy sources for one model atmosphere with cirrus clouds between 8 and 12 km. The large excursions of $\dot{e}^r (= -\dot{e}^a)$ are a well-known phenomenon, with strong cooling at cloud tops and strong heating at cloud bases. There are accompanying positive and negative excursions in $\dot{s}^r$ and So. These excursions show a great deal of cancelation, unlike the data in Fig.3. $\dot{s}^r_{irr}$ is constrained to be positive, and its variability is less than that of the other sources. If it is assumed to be constant, or to be weakly correlated with the solar flux, a rough estimate of the atmospheric entropy source may be made.

7 Conclusions

Sources of atmospheric entropy can provide important information for climate calculations. For cloudless skies the integrated source of atmospheric entropy can be inferred, to a useful level of accuracy, from the outgoing flux of radiation

Appendix: Approximate equation of transfer for entropy radiance

For a stratified atmosphere, Eq.(23) may be written,

$$\xi \frac{dI_\nu}{d\tau_\nu} = I_\nu - B_\nu(T), \tag{36}$$

where ξ is the cosine of the zenith angle, and τ_ν is the optical depth. In local thermodynamic equilibrium, B_ν, the source function) is equal to the Planck function.

From equations (14) and (36) wc may write an equation of transfer for the entropy radiance,

$$\xi \frac{dL_\nu}{d\tau_\nu} = L_\nu - S_\nu , \tag{37}$$

where,

$$S_\nu = \frac{2k\nu^2}{c^2}\left\{\left(\frac{c^2B_\nu}{2h\nu^3}+1\right)\ln\left(\frac{c^2I_\nu}{2h\nu^3}+1\right) - \frac{c^2B_\nu}{2h\nu^3}\ln\left(\frac{c^2I_\nu}{2h\nu^3}\right)\right\}, \tag{38}$$

is, by analogy with ***(36),*** the entropy source function.

Essex (1984) argues that ***(38)*** may be approximated by setting $I_\nu = B_\nu$ *in* this term but not, of course, in equation (14). The attraction of this assumption is that the source function becomes a function of T and ν only, and (37) may be solved by the same methods that are used to solve (36). Essex refers to this assumption as the Equal Thermodynamic Protocol refinement of local thermodynamic equilibrium (ETP-LTE), ancl shows that it is justified in the limit $h\nu/kT \gg 1$.

For our purposes, the radiation entropy must bc integrated over all frequencies, and the above inequality dots not hold over the full range of integration. However, Essex and Lesins (1 992) and Li, Chýlek, and Lesins (1 994) have applied ETP-LTE to the calculation of integrated entropy fluxes.

A detailed numerical assessment of this method would bc valuable, for it is useful, if applicable. Wc shall look at one situation for whit.11 exact and approximate solutions arc easy to obtain. Wc calculate the downward radiaucc from a grey-absorbing isothermal slab of optical depth, τ_ν, with no radiation incident from above. This is probably the most severe test of ETP-LTE that could bc devised, and other situations, including non-grey atmospheres, may have smaller errors.

The solution to (36) is,

$$I_\nu(\tau_\nu/\xi) = B_\nu\left(1 - e^{\tau_\nu/\xi}\right) \tag{39}$$

For downward directed radiances, ξ is negative

If we write (14) in the operational form,

$$L_\nu = L_\nu [I_\nu] , \qquad (40)$$

then the exact solution for the entropy radiance can be obtained from Eqs. (39) and (40),

$$L_\nu^{\text{exact}}(\tau_\nu/\xi) = L_\nu \left[B_\nu (1 - e^{\tau_\nu/\xi}) \right] \qquad (41)$$

From (14), (38), and (40), the approximate source function is,

$$S_\nu = L_\nu [B_\nu] , \qquad (42)$$

and the ETP-LTE solution becomes,

$$L_\nu^{\text{approx}}(\tau_\nu/\xi) = L_\nu [B_\nu] \left(1 - e^{\tau_\nu/\xi} \right) . \qquad (43)$$

For grey absorption, $\tau_\nu = \tau$, and (42) and (43) may be integrated over all frequencies to give,

$$L^{\text{approx}}(u) = \frac{4}{3} \frac{\sigma}{\pi} T^3 u, \qquad (44)$$

$$L^{\text{exact}}(u) = \frac{4}{3} \frac{\sigma}{\pi} T^3 \chi(u), \qquad (45)$$

where $u = 1 - e^{\tau/\xi}$. The function $\chi(u)$ has most recently been evaluated by Stephens and O'Brien (1993). For $u < 0.1$,

$$\chi(u) = u(0.96516 - 0.27765 \ln u). \qquad (46)$$

$\chi(u)$ for larger u is shown in Fig.5. For large optical depths ($u \to 1$), ETP-LTE is satisfactory, but for small optical depths ($u \to 0$), the fractional errors can be very large. The radiation-to-space result, discussed in §5, implies that atmospheric energy and entropy sources are associated with optical depths that are not large.

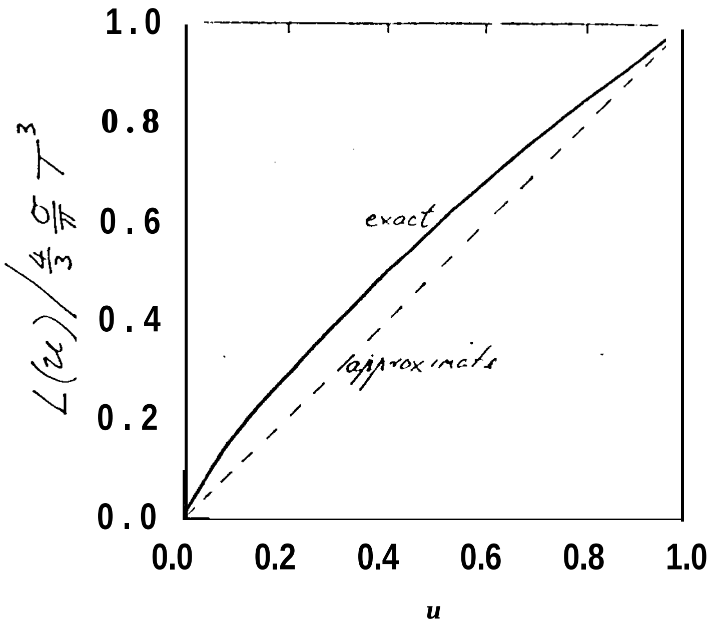

Figure 5: $L(u)/\frac{4}{3}\frac{\sigma}{\pi}T^3$ *as* a function of u from Eqs.(44) and (45).

References

Chahine, M.T.	1977	Remote sensing of cloudy atmospheres, 11. Multiple cloud formations, *J. Atmos. Sci.*, 34, 744-757
Essex, C.	1984	Minimum entropy production in the steady state and radiative transfer, *Astrophys. J.*, 285, 279-293
Essex, C. and Lesins, G.B.	1992	Radiative transfer and equal thermodynamic protocol, *IRS'92: Current problems in atmospheric radiation.* Deepak, Hampton, VA, 513-515
Goody, R.M and Yung, Y. L.	1989	*Atmospheric radiation, 2nd. Edition.* Oxford University Press, New York
Lesins, G. B.	1990	On the relationship between radiative entropy and temperature distribution, *J. Atmos. Sci., 47, 795-803*
Li, J., Chýlek, P., and Lesins, G. B.	1994	Entropy in climate models. Part 1: Vertical structure of atmospheric entropy production, *J. Atmos. Sci.*, **51**, 1691-1701
Paltridge, G.W.	1975	Global dynamics and climate a system of minimum entropy exchange, Q. *J.* R. *Meteorol. Sot., 101, 475-484*
Peixoto, J.P., Oort, A.H., Almeida, M. and Tomé, A	1991	Entropy budget of the atmosphere, *J. Geophys. Res.*, 96, 10981-10988
Planck, M.	1955	*The theory of heat radiation.* Dover Publications, New York, 190
Rosen, P.	1954	Entropy of radiation, *Phys. Rev.*, **96**, 555-556
Stephens, G.L. and O'Brien, D.M.	1993	Entropy and climate: ERBE observations of the entropy production of the earth, Q. *J. R. Meteorol. Sot.* 119, 121-152